9.4 NORTON'S THEOREM

In Section 8.3, we learned that every voltage source with a series internal resistance has a current source equivalent. The current source equivalent can be determined by **Norton's theorem** (Fig. 9.58). It can also be found through the conversions of Section 8.3.

The theorem states the following:

Any two-terminal linear bilateral dc network can be replaced by an equivalent circuit consisting of a current source and a parallel resistor, as shown in Fig. 9.59.

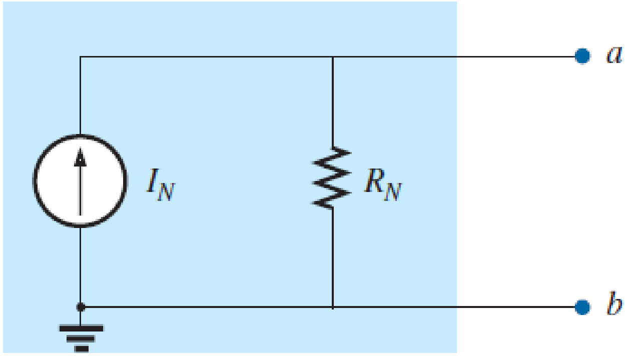


FIG. 9.59
Norton equivalent circuit.

The discussion of Thévenin's theorem with respect to the equivalent circuit can also be applied to the Norton equivalent circuit. The steps leading to the proper values of I_N and R_N are now listed.

Norton's Theorem Procedure

Preliminary:

1. *Remove that portion of the network across which the Norton equivalent circuit is found.*
2. *Mark the terminals of the remaining two-terminal network.*

R_N:

3. *Calculate R_N by first setting all sources to zero (voltage sources are replaced with short circuits, and current sources with open circuits) and then finding the resultant resistance between the two*

marked terminals. (If the internal resistance of the voltage and/or current sources is included in the original network, it must remain when the sources are set to zero.) Since $R_N = R_{Th}$, the procedure and value obtained using the approach described for Thévenin's theorem will determine the proper value of R_N.

I_N:

4. *Calculate I_N by first returning all sources to their original position and then finding the short-circuit current between the marked terminals. It is the same current that would be measured by an ammeter placed between the marked terminals.*

Conclusion:

5. *Draw the Norton equivalent circuit with the portion of the circuit previously removed replaced between the terminals of the equivalent circuit.*

The Norton and Thévenin equivalent circuits can also be found from each other by using the source transformation discussed earlier in this chapter and reproduced in Fig. 9.60.

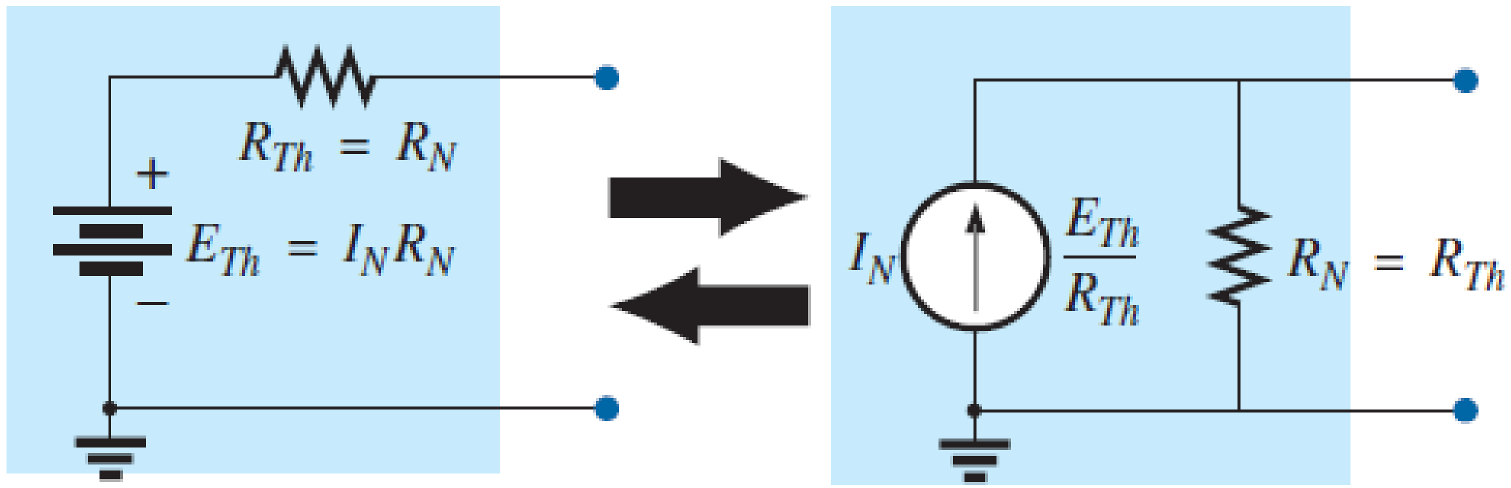
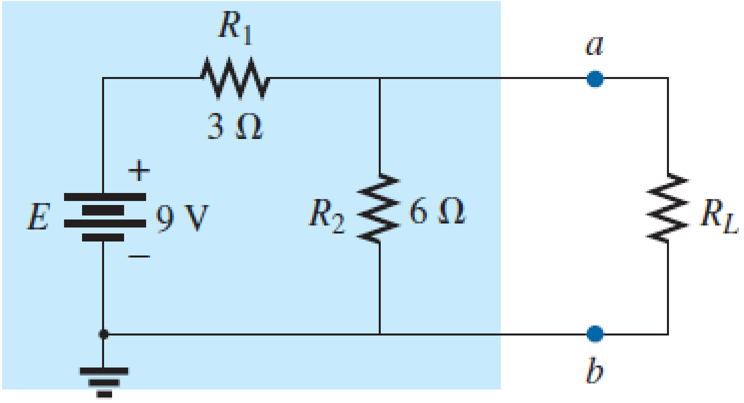


FIG. 9.60
Converting between Thévenin and Norton equivalent circuits.

EXAMPLE 9.11 Find the Norton equivalent circuit for the network in the shaded area in Fig. 9.61.



Solution:

Steps 1 and 2: See Fig. 9.62.

Step 3: See Fig. 9.63, and

$$R_N = R_1 \| R_2 = 3\ \Omega \| 6\ \Omega = \frac{(3\ \Omega)(6\ \Omega)}{3\ \Omega + 6\ \Omega} = \frac{18\ \Omega}{9} = \mathbf{2\ \Omega}$$

Step 4: See Fig. 9.64, which clearly indicates that the short-circuit connection between terminals *a* and *b* is in parallel with R_2 and eliminates its effect. I_N is therefore the same as through R_1, and the full battery voltage appears across R_1 since

$$V_2 = I_2R_2 = (0)6\ \Omega = 0\ \text{V}$$

Therefore,

$$I_N = \frac{E}{R_1} = \frac{9\ \text{V}}{3\ \Omega} = \mathbf{3\ A}$$

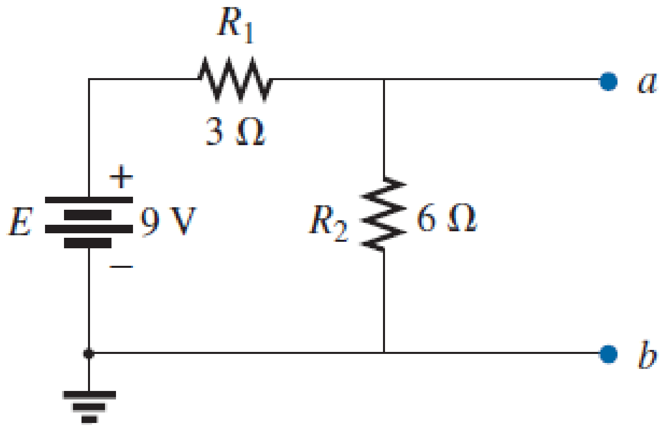


FIG. 9.62

Identifying the terminals of particular interest for the network in Fig. 9.61.

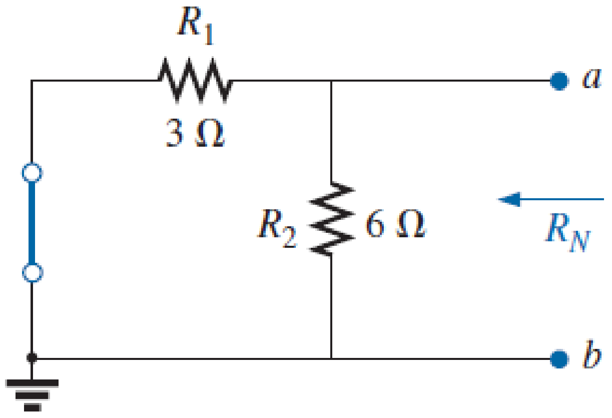


FIG. 9.63

Determining R_N for the network in Fig. 9.62.

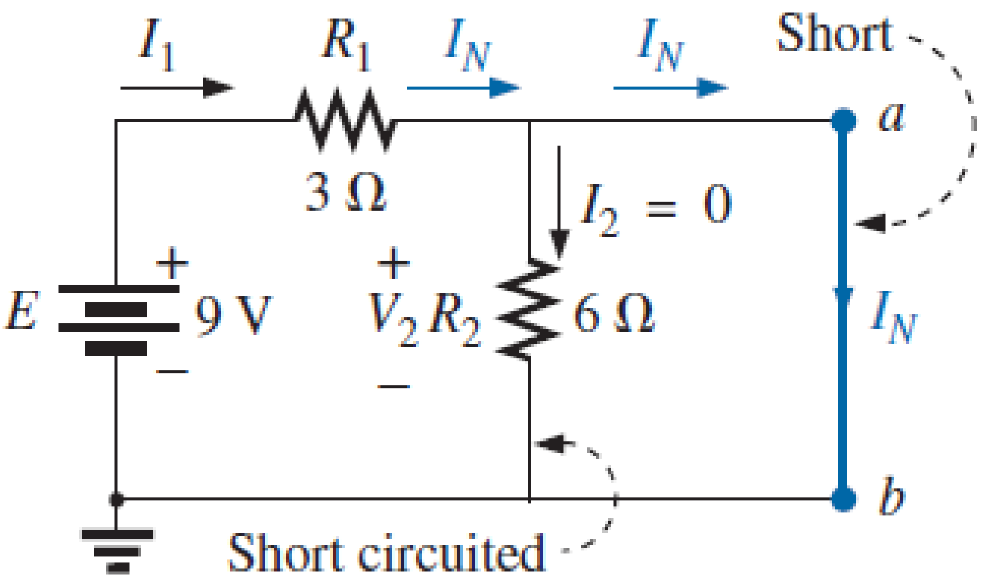


FIG. 9.64

Determining I_N for the network in Fig. 9.62.

$$V_2 = I_2R_2 = (0)6\ \Omega = 0\ \text{V}$$

$$I_N = \frac{E}{R_1} = \frac{9\ \text{V}}{3\ \Omega} = \mathbf{3\ A}$$

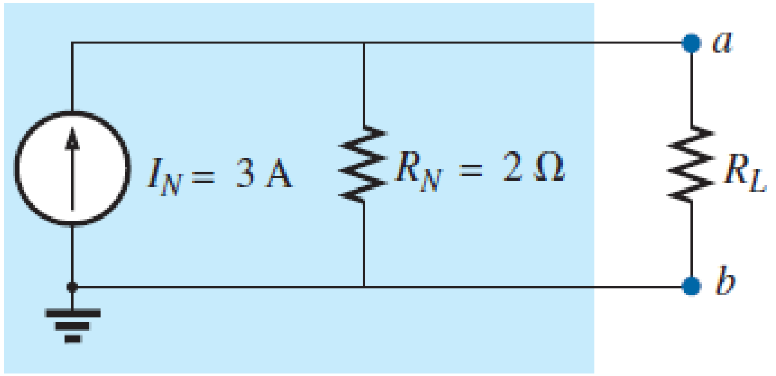


FIG. 9.65

Substituting the Norton equivalent circuit for the network external to the resistor R_L in Fig. 9.61.

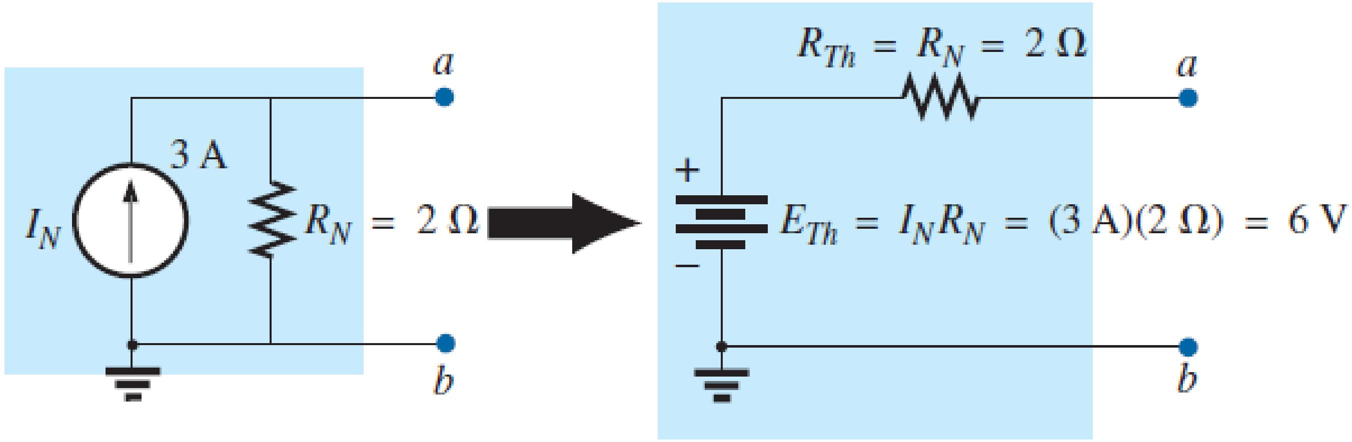


FIG. 9.66

Converting the Norton equivalent circuit in Fig. 9.65 to a Thévenin equivalent circuit.

EXAMPLE 9.12 Find the Norton equivalent circuit for the network external to the 9 Ω resistor in Fig. 9.67.

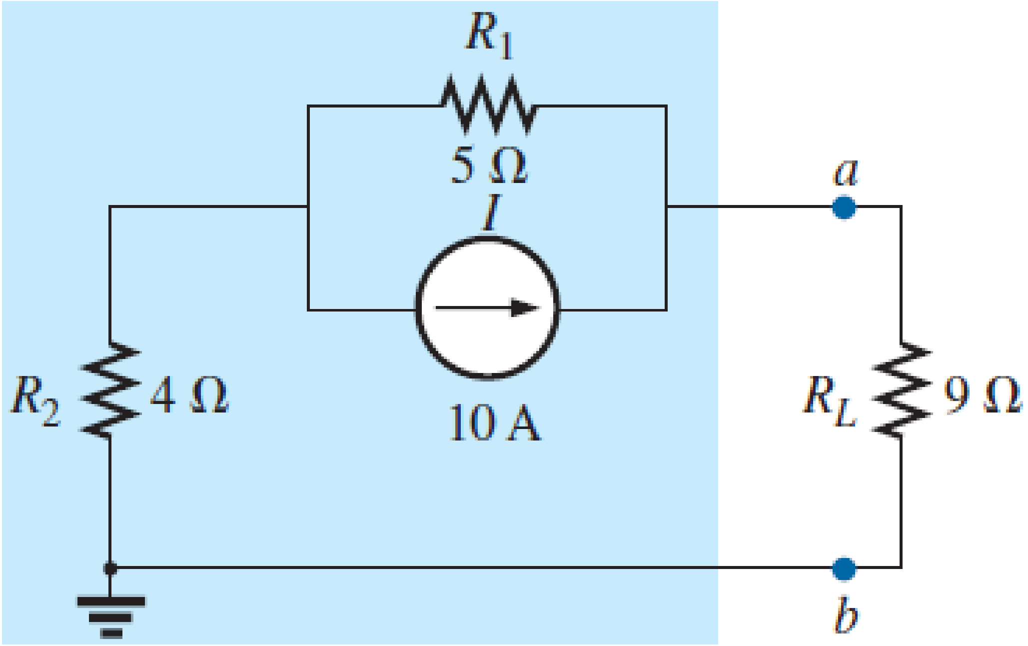


FIG. 9.67
Example 9.12.

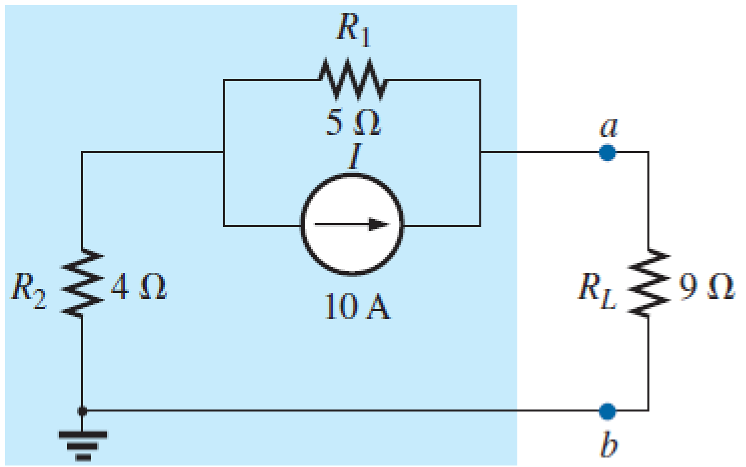


FIG. 9.67
Example 9.12.

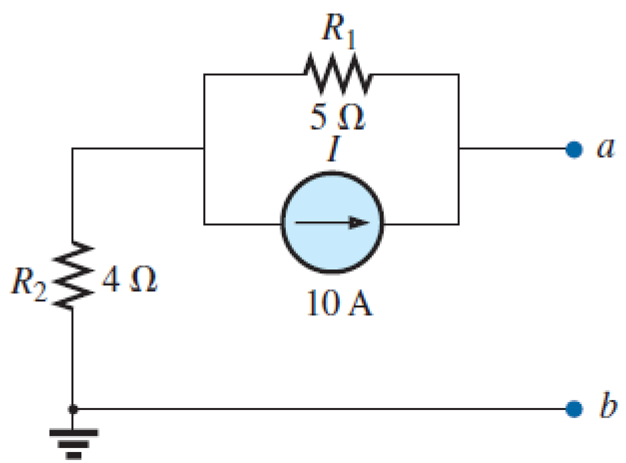


FIG. 9.68
Identifying the terminals of particular interest for the network in Fig. 9.67.

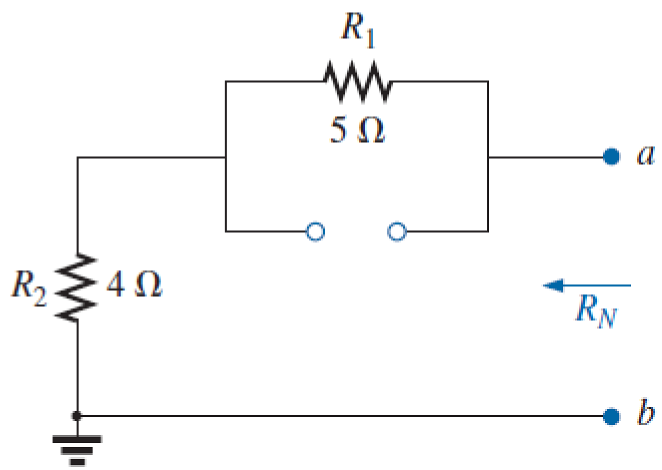


FIG. 9.69
Determining R_N for the network in Fig. 9.68.

$$R_N = R_1 + R_2 = 5\ \Omega + 4\ \Omega = 9\ \Omega$$

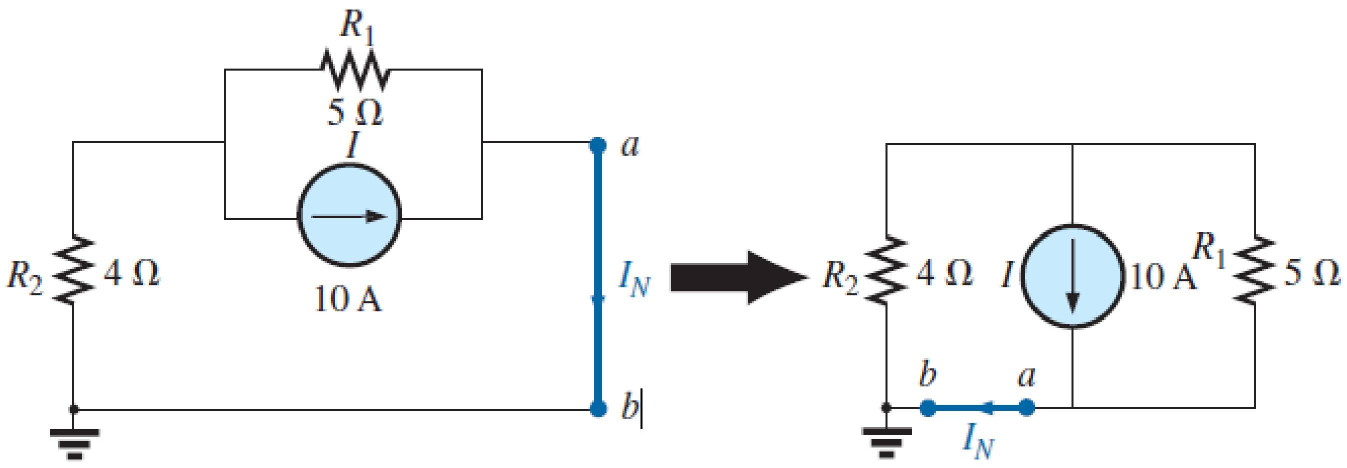


FIG. 9.70
Determining I_N for the network in Fig. 9.68.

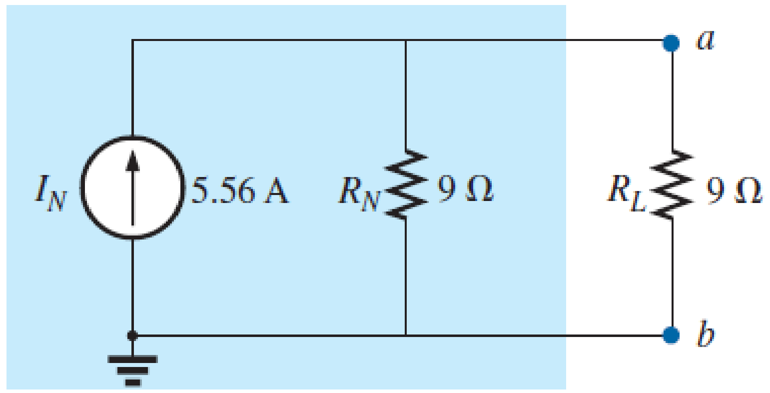


FIG. 9.71
Substituting the Norton equivalent circuit for the network external to the resistor R_L in Fig. 9.67.

Step 4: As shown in Fig. 9.70, the Norton current is the same as the current through the 4 Ω resistor. Applying the current divider rule,

$$I_N = \frac{R_1 I}{R_1 + R_2} = \frac{(5\ \Omega)(10\ \text{A})}{5\ \Omega + 4\ \Omega} = \frac{50\ \text{A}}{9} = \mathbf{5.56\ A}$$

EXAMPLE 9.13 (Two sources) Find the Norton equivalent circuit for the portion of the network to the left of *a*-*b* in Fig. 9.72.

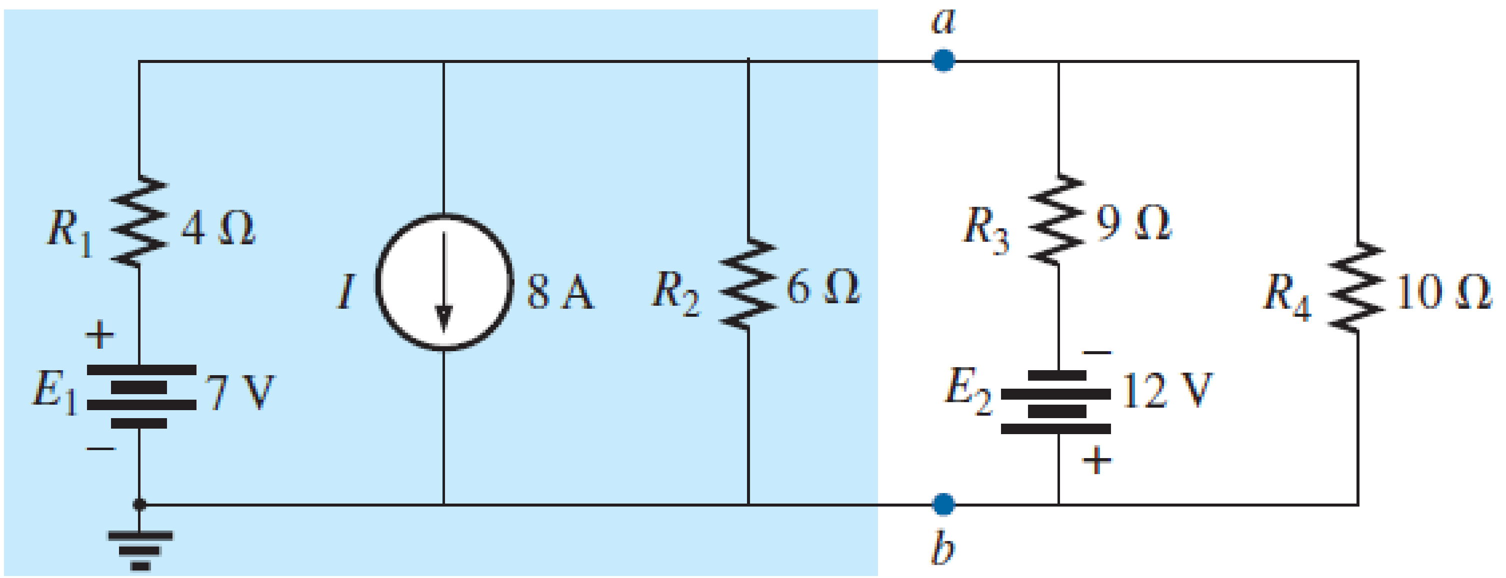


FIG. 9.72

Example 9.13.

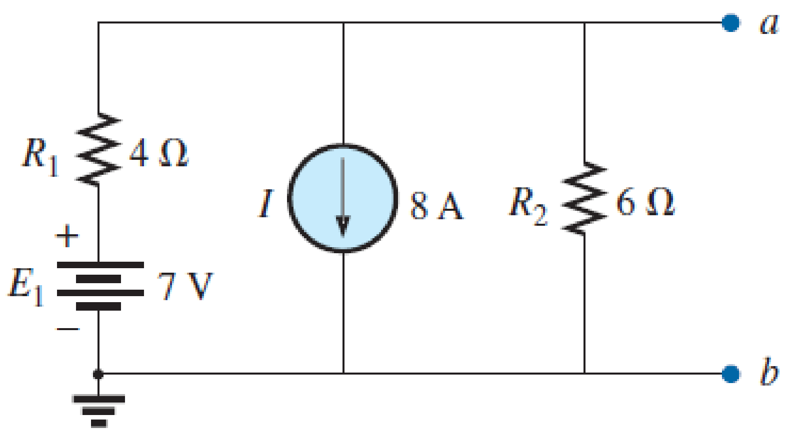
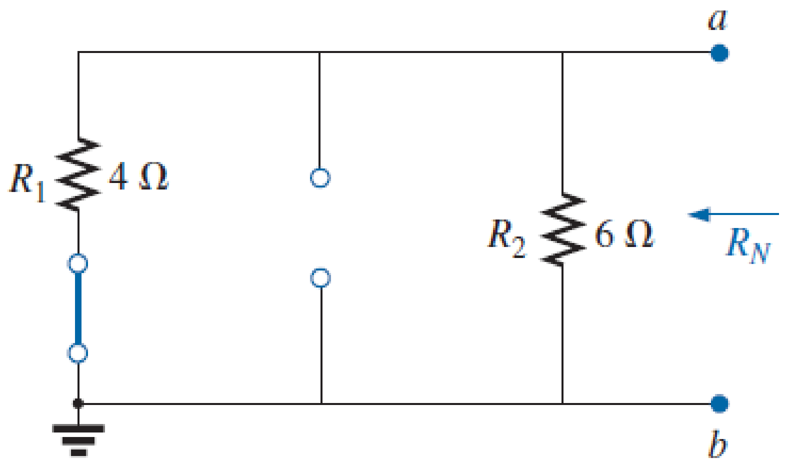


FIG. 9.73

Identifying the terminals of particular interest for the network in Fig. 9.72.



$$R_N = R_1 \| R_2 = 4\ \Omega \| 6\ \Omega = \frac{(4\ \Omega)(6\ \Omega)}{4\ \Omega + 6\ \Omega} = \frac{24\ \Omega}{10} = \mathbf{2.4\ \Omega}$$

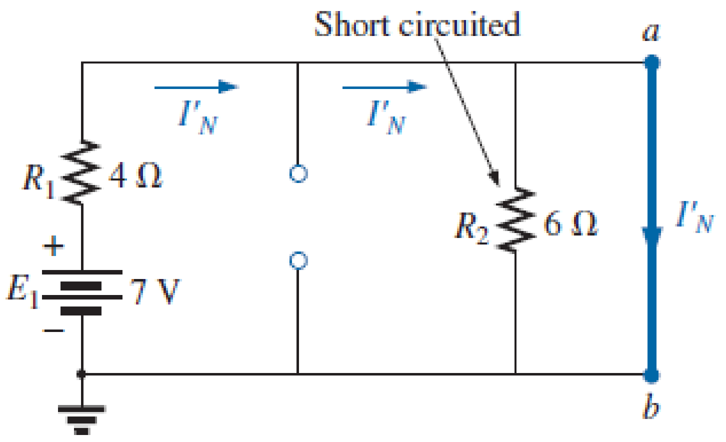


FIG. 9.75

Determining the contribution to I_N from the voltage source E_1.

Step 4: (Using superposition) For the 7 V battery (Fig. 9.75),

$$I'_N = \frac{E_1}{R_1} = \frac{7\text{ V}}{4\ \Omega} = 1.75\text{ A}$$

For the 8 A source (Fig. 9.76), we find that both R_1 and R_2 have been "short circuited" by the direct connection between *a* and *b*, and

$$I''_N = I = 8\text{ A}$$

The result is

$$I_N = I''_N - I'_N = 8\text{ A} - 1.75\text{ A} = \mathbf{6.25\ A}$$

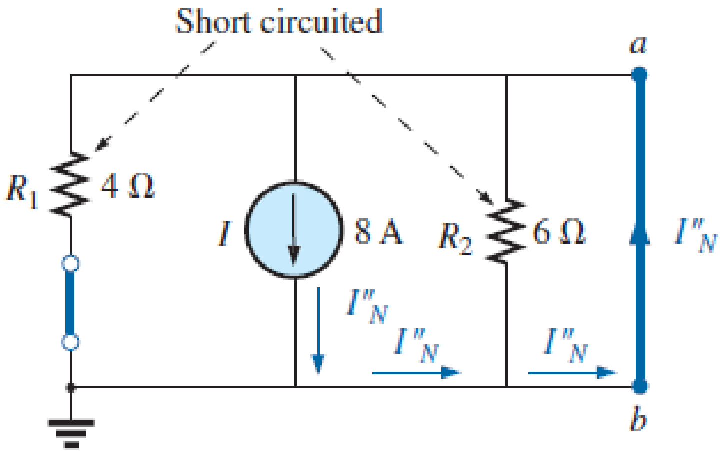


FIG. 9.76

Determining the contribution to I_N from the current source I.

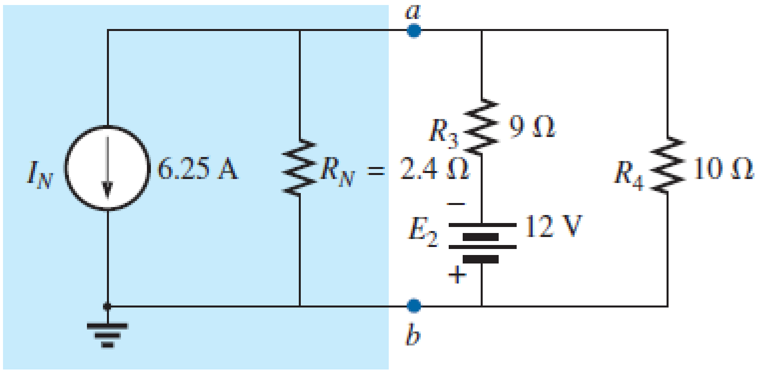


FIG. 9.77

Substituting the Norton equivalent circuit for the network to the left of terminals a-b in Fig. 9.72.

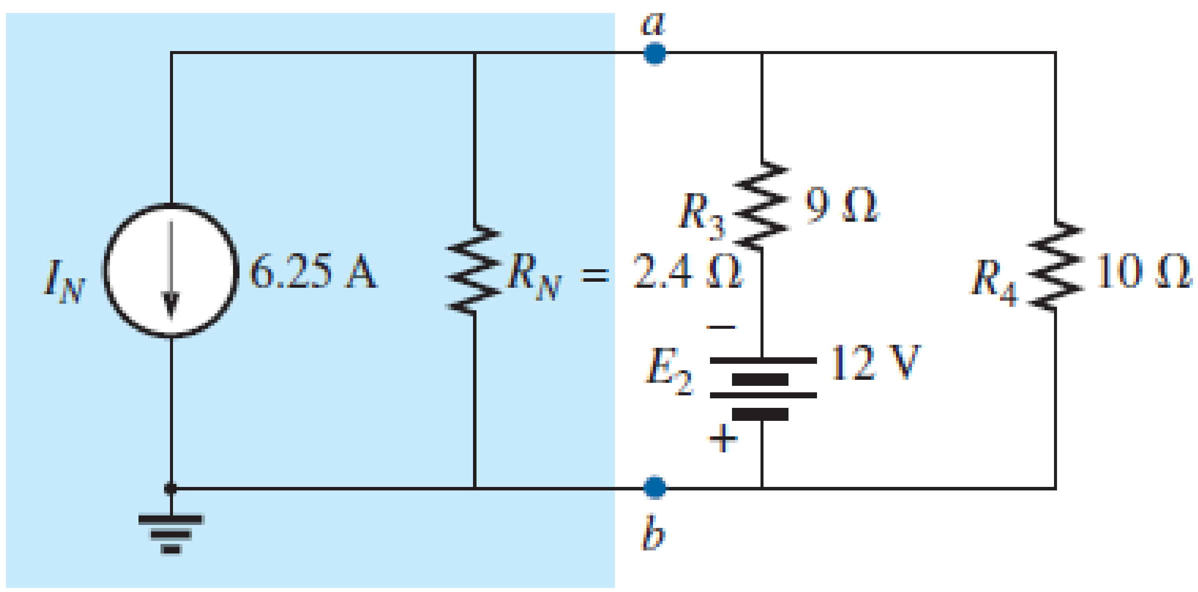


FIG. 9.77

Substituting the Norton equivalent circuit for the network to the left of terminals a-b in Fig. 9.72.

Using Norton's theorem, find R_N and I_N of the circuit in Fig. 4.43 at terminals *a*-*b*.

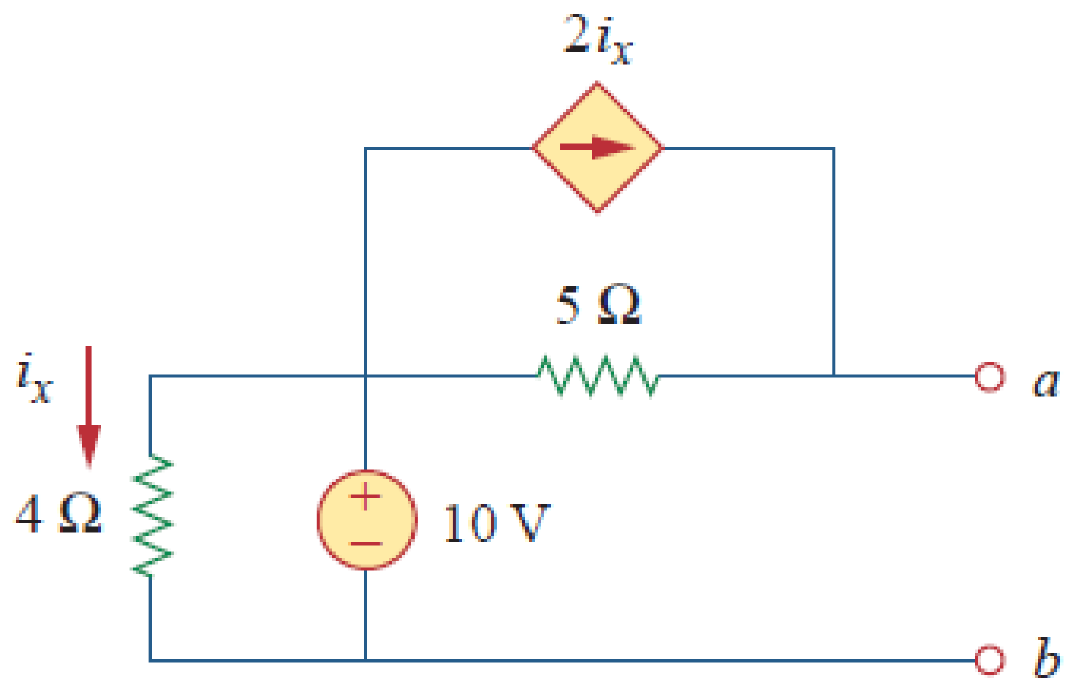


Figure 4.43
For Example 4.12.

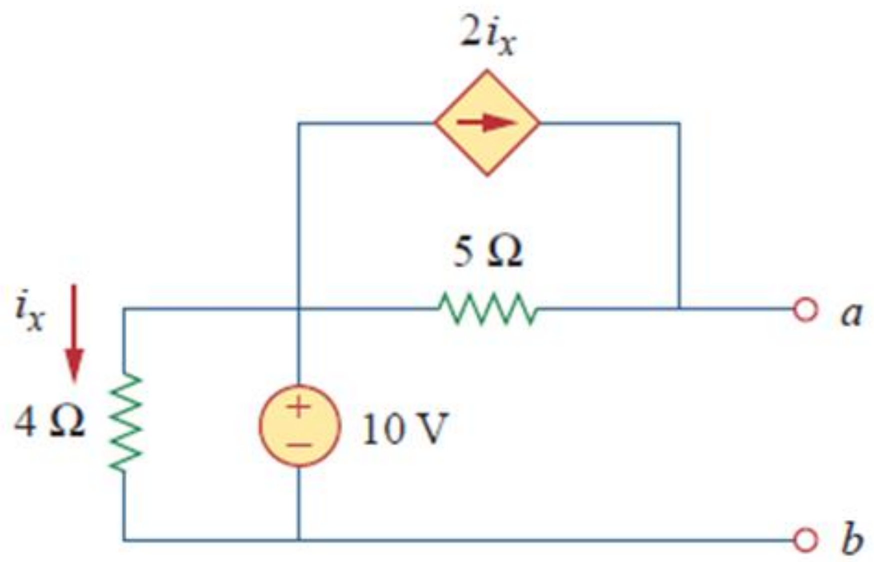
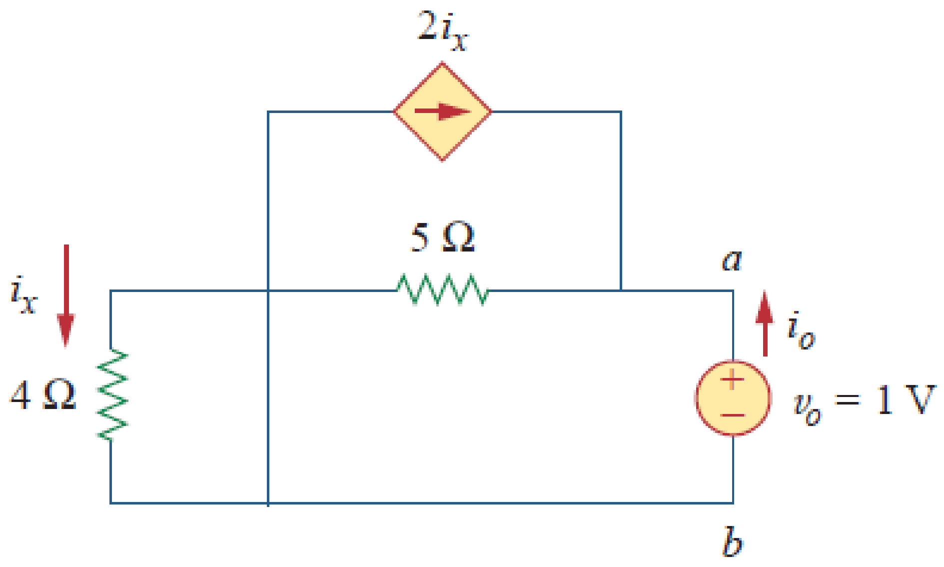
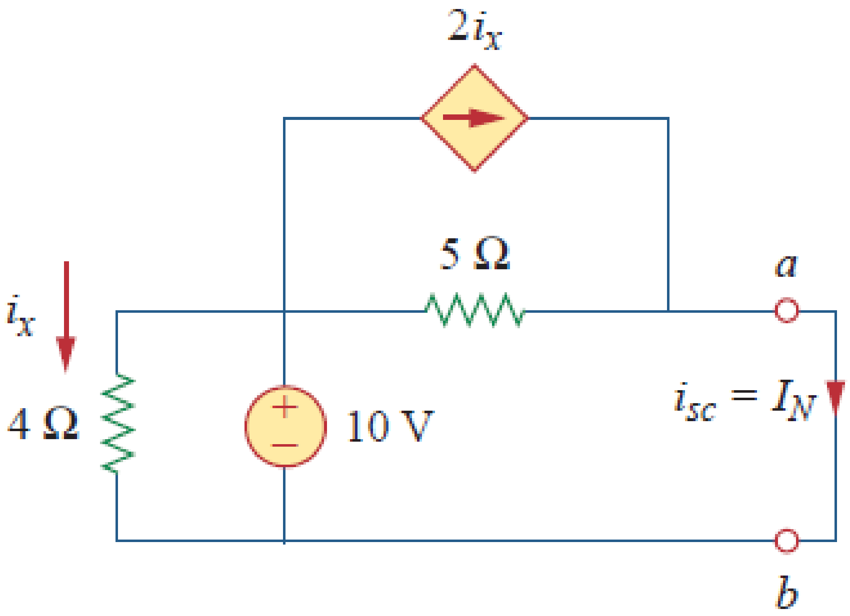


Figure 4.43
For Example 4.12.



To find R_N, we set the independent voltage source equal to zero and connect a voltage source of $v_o = 1$ V (or any unspecified voltage v_o) to the terminals. We obtain the circuit in Fig. 4.44(a). We ignore the 4-Ω resistor because it is short-circuited. Also due to the short circuit, the 5-Ω resistor, the voltage source, and the dependent current source are all in parallel. Hence, $i_x = 0$. At node a, $i_o = \frac{1v}{5\Omega} = 0.2$ A, and

$$R_N = \frac{v_o}{i_o} = \frac{1}{0.2} = 5\ \Omega$$



To find I_N, we short-circuit terminals a and b and find the current i_{sc}, as indicated in Fig. 4.44(b). Note from this figure that the 4-Ω resistor, the 10-V voltage source, the 5-Ω resistor, and the dependent current source are all in parallel. Hence,

$$i_x = \frac{10}{4} = 2.5 \text{ A}$$

At node a, KCL gives

$$i_{sc} = \frac{10}{5} + 2i_x = 2 + 2(2.5) = 7 \text{ A}$$

Thus,

$$I_N = 7 \text{ A}$$

Experimental Procedure

The Norton current is measured in the same way as described for the short-circuit current (I_{sc}) for the Thévenin network. Since the Norton and Thévenin resistances are the same, the same procedures can be followed as described for the Thévenin network.

9.5 MAXIMUM POWER TRANSFER THEOREM

When designing a circuit, it is often important to be able to answer one of the following questions:

What load should be applied to a system to ensure that the load is receiving maximum power from the system?

Fortunately, the process of finding the load that will receive maximum power from a particular system is quite straightforward due to the **maximum power transfer theorem,** which states the following:

A load will receive maximum power from a network when its resistance is exactly equal to the Thévenin resistance of the network applied to the load. That is,

$$R_L = R_{Th} \tag{9.2}$$

In other words, for the Thévenin equivalent circuit in Fig. 9.78, when the load is set equal to the Thévenin resistance, the load will receive maximum power from the network.

Using Fig. 9.78 with $R_L = R_{Th}$, the maximum power delivered to the load can be determined by first finding the current:

$$I_L = \frac{E_{Th}}{R_{Th} + R_L} = \frac{E_{Th}}{R_{Th} + R_{Th}} = \frac{E_{Th}}{2R_{Th}}$$

Then substitute into the power equation:

$$P_L = I_L^2 R_L = \left(\frac{E_{Th}}{2R_{Th}}\right)^2 (R_{Th}) = \frac{E_{Th}^2 R_{Th}}{4R_{Th}^2}$$

and

$$P_{L_{max}} = \frac{E_{Th}^2}{4R_{Th}} \tag{9.3}$$

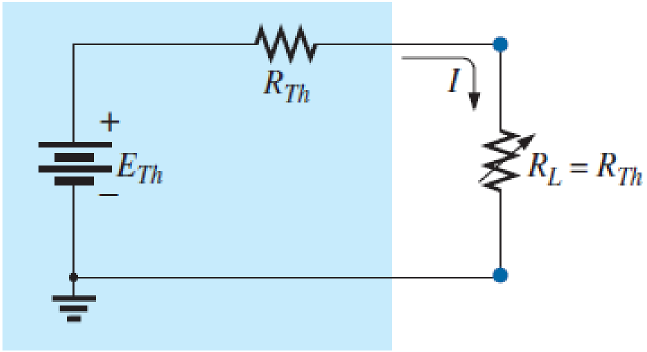


FIG. 9.78
Defining the conditions for maximum power to a load using the Thévenin equivalent circuit.

If the load applied is less than the Thévenin resistance, the power to the load will drop off rapidly as it gets smaller. However, if the applied load is greater than the Thévenin resistance, the power to the load will not drop off as rapidly as it increases.

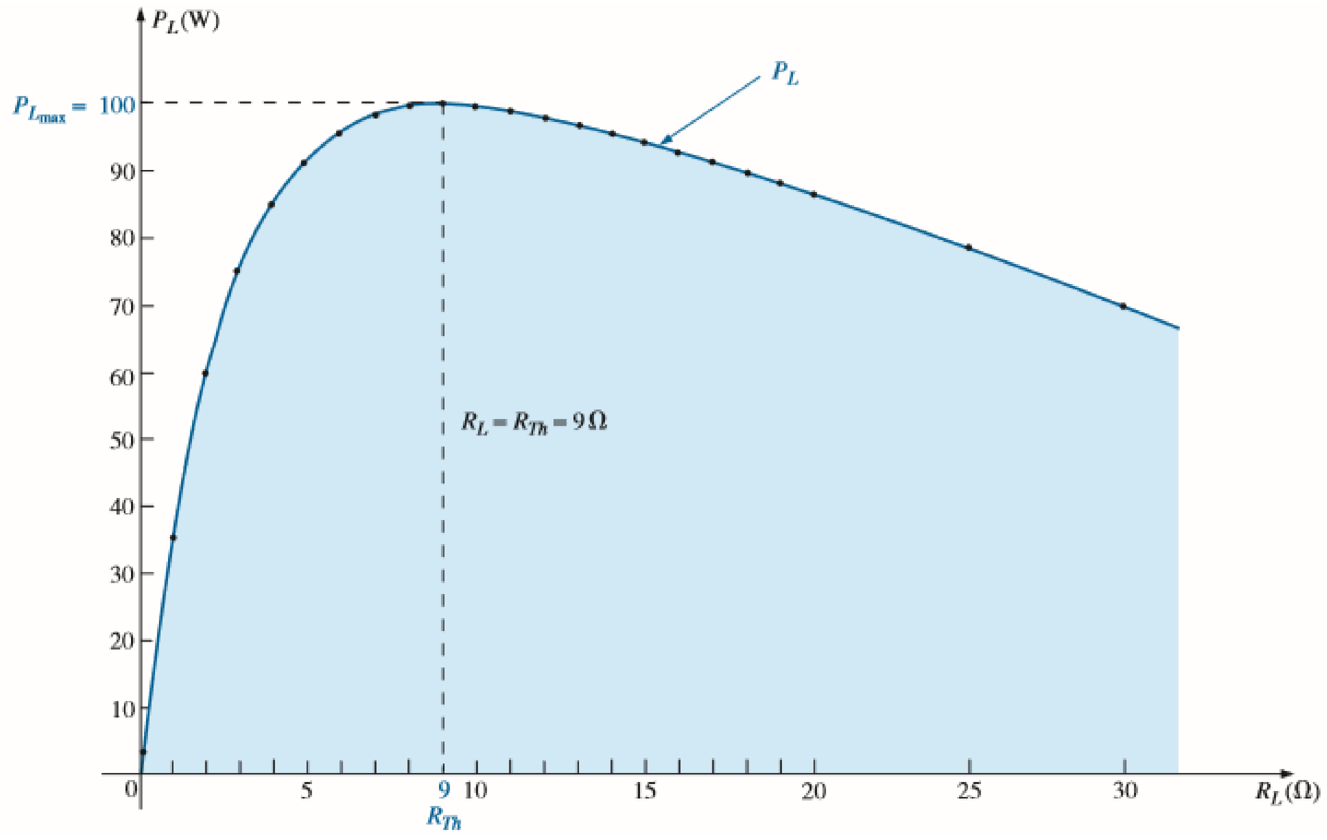


FIG. 9.80

P_L versus R_L for the network in Fig. 9.79.

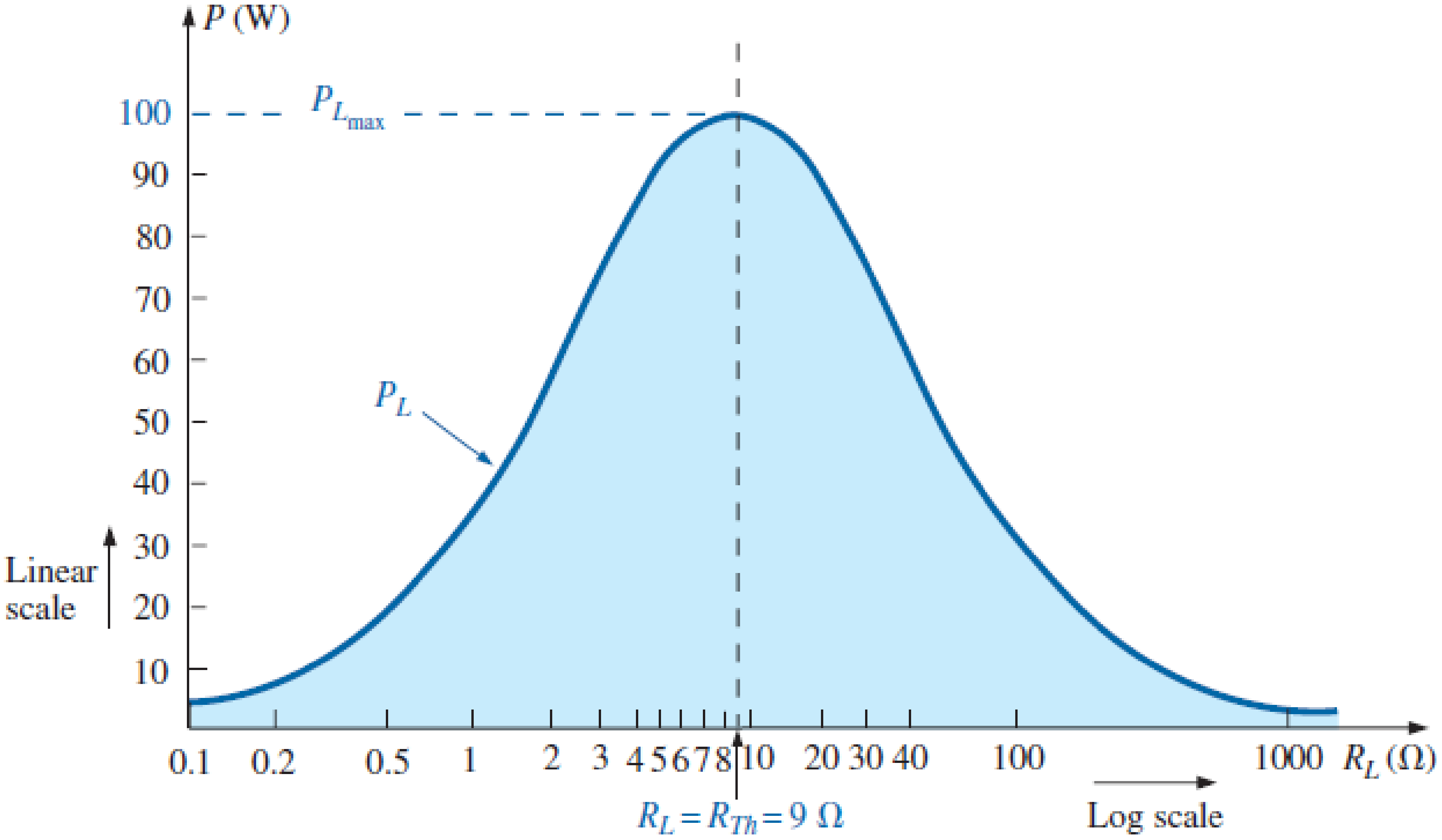


FIG. 9.81

P_L versus R_L for the network in Fig. 9.79.

The total power delivered by a supply such as E_{Th} is absorbed by both the Thévenin equivalent resistance and the load resistance. Any power delivered by the source that does not get to the load is lost to the Thévenin resistance.

Under maximum power conditions, only half the power delivered by the source gets to the load. Now, that sounds disastrous, but remember that we are starting out with a fixed Thévenin voltage and resistance, and the above simply tells us that we must make the two resistance levels equal if we want maximum power to the load. On an efficiency basis, we are working at only a 50% level, but we are content because *we are getting maximum power out of our system.*

The dc operating efficiency is defined as the ratio of the power delivered to the load (P_L) to the power delivered by the source (P_s). That is,

$$\eta\% = \frac{P_L}{P_s} \times 100\% \tag{9.4}$$

For the situation where $R_L = R_{Th}$,

$$\eta\% = \frac{I_L^2 R_L}{I_L^2 R_T} \times 100\% = \frac{R_L}{R_T} \times 100\% = \frac{R_{Th}}{R_{Th} + R_{Th}} \times 100\%$$

$$= \frac{R_{Th}}{2R_{Th}} \times 100\% = \frac{1}{2} \times 100\% = \mathbf{50\%}$$

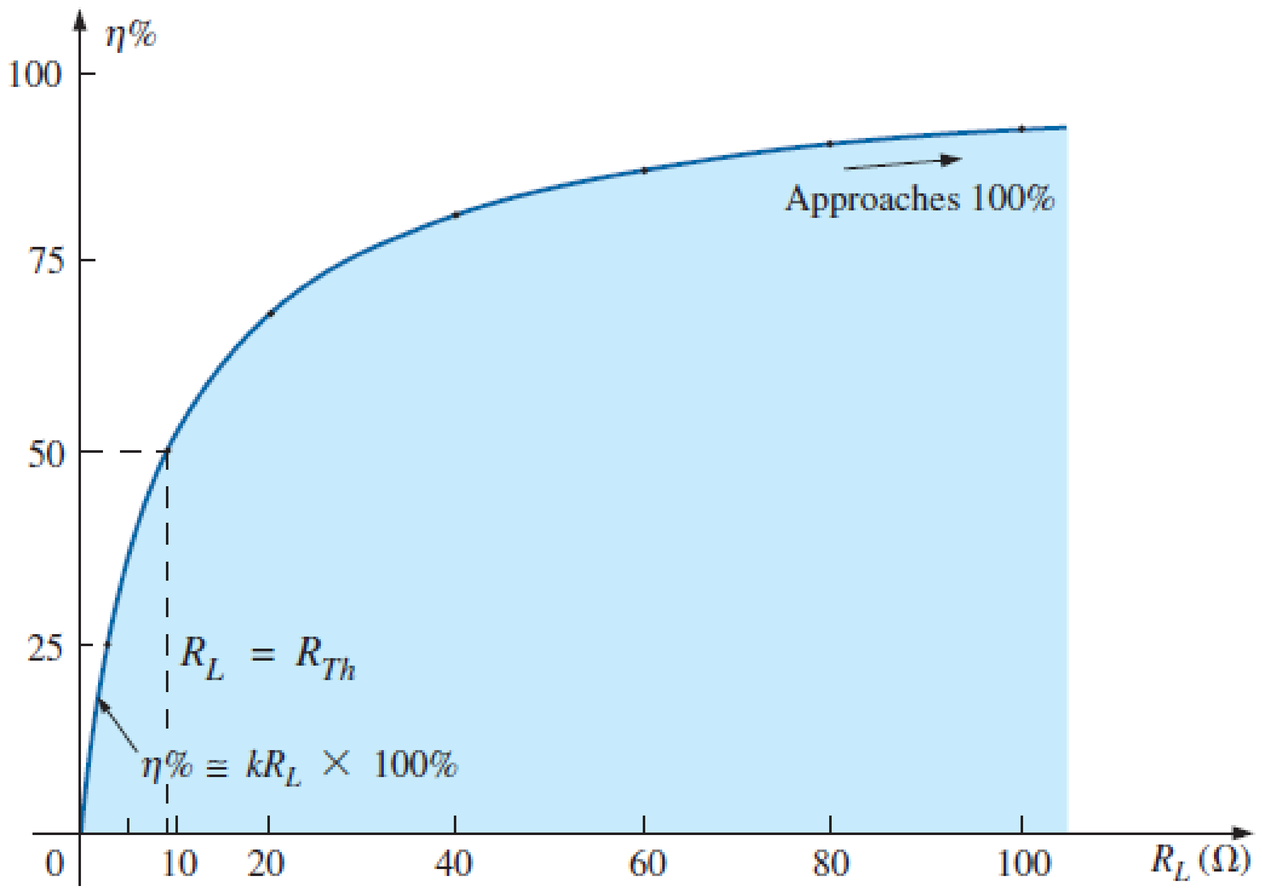


FIG. 9.82

Efficiency of operation versus increasing values of R_L.

For the circuit in Fig. 9.79, if we plot the efficiency of operation versus load resistance, we obtain the plot in Fig. 9.82, which clearly shows that the efficiency continues to rise to a 100% level as R_L gets larger. Note in particular that the efficiency is 50% when $R_L = R_{Th}$.

To ensure that you completely understand the effect of the maximum power transfer theorem and the efficiency criteria, consider the circuit in Fig. 9.83 where the load resistance is set at 100 Ω and the power to the Thévenin resistance and to the load are calculated as follows:

$$I_L = \frac{E_{Th}}{R_{Th} + R_L} = \frac{60\text{ V}}{9\ \Omega + 100\ \Omega} = \frac{60\text{ V}}{109\ \Omega} = 550.5\text{ mA}$$

with $$P_{R_{Th}} = I_L^2 R_{Th} = (550.5\text{ mA})^2(9\ \Omega) \cong \mathbf{2.73\ W}$$

and $$P_L = I_L^2 R_L = (550.5\text{ mA})^2(100\ \Omega) \cong \mathbf{30.3\ W}$$

The results clearly show that most of the power supplied by the battery is getting to the load—a desirable attribute on an efficiency basis. However, the power getting to the load is only 30.3 W compared to the 100 W obtained under maximum power conditions. In general, therefore, the following guidelines apply:

If efficiency is the overriding factor, then the load should be much larger than the internal resistance of the supply. If maximum power transfer is desired and efficiency less of a concern, then the conditions dictated by the maximum power transfer theorem should be applied.

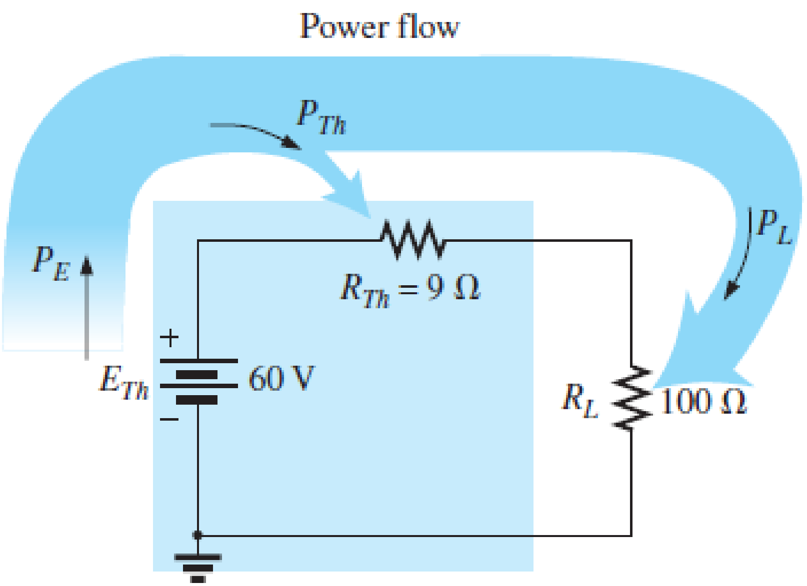


FIG. 9.83
Examining a circuit with high efficiency but a relatively low level of power to the load.

In all of the above discussions, the effect of changing the load was discussed for a fixed Thévenin resistance. Looking at the situation from a different viewpoint,

if the load resistance is fixed and does not match the applied Thévenin equivalent resistance, then some effort should be made (if possible) to redesign the system so that the Thévenin equivalent resistance is closer to the fixed applied load.

In other words, if a designer faces a situation where the load resistance is fixed, he/she should investigate whether the supply section should be replaced or redesigned to create a closer match of resistance levels to produce higher levels of power to the load.

For the Norton equivalent circuit in Fig. 9.84, maximum power will be delivered to the load when

$$R_L = R_N \tag{9.5}$$

This result [Eq. (9.5)] will be used to its fullest advantage in the analysis of transistor networks, where the most frequently applied transistor circuit model uses a current source rather than a voltage source.

For the Norton circuit in Fig. 9.84,

$$P_{L_{\max}} = \frac{I_N^2 R_N}{4} \quad \text{(W)} \tag{9.6}$$

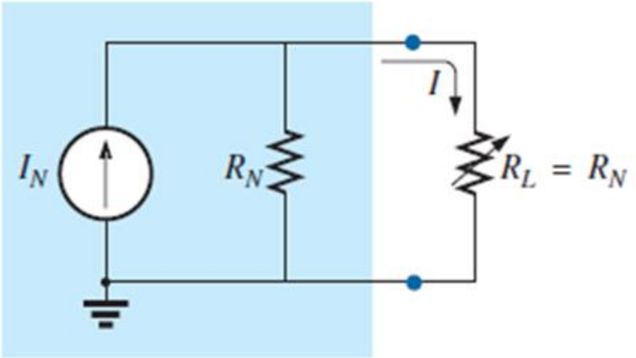


FIG. 9.84
Defining the conditions for maximum power to a load using the Norton equivalent circuit.

Find the value of R_L for maximum power transfer in the circuit of Fig. 4.50. Find the maximum power.

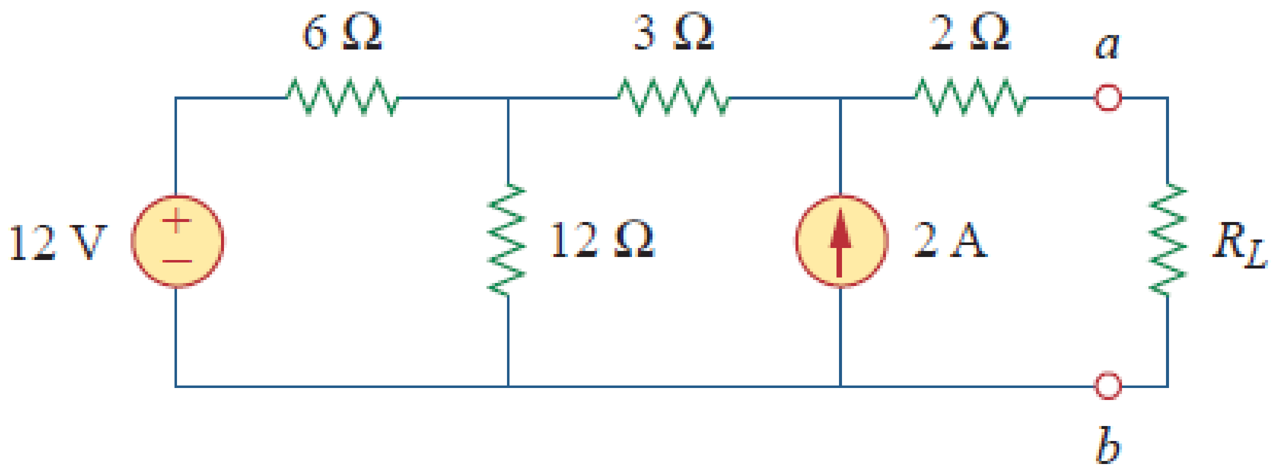
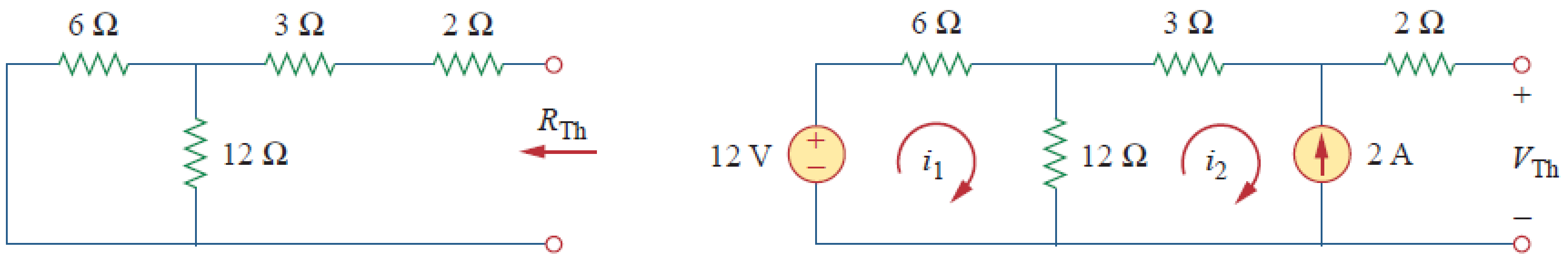


Figure 4.50
For Example 4.13.

We need to find the Thevenin resistance R_{Th} and the Thevenin voltage V_{Th} across the terminals a-b. To get R_{Th}, we use the circuit in Fig. 4.51(a) and obtain

$$R_{Th} = 2 + 3 + 6 \| 12 = 5 + \frac{6 \times 12}{18} = 9\ \Omega$$



(a) (b)

Figure 4.51
For Example 4.13: (a) finding R_{Th}, (b) finding V_{Th}.

To get V_{Th}, we consider the circuit in Fig. 4.51(b). Applying mesh analysis gives

$$-12 + 18i_1 - 12i_2 = 0, \qquad i_2 = -2\text{ A}$$

Solving for i_1, we get $i_1 = -2/3$. Applying KVL around the outer loop to get V_{Th} across terminals *a*-*b*, we obtain

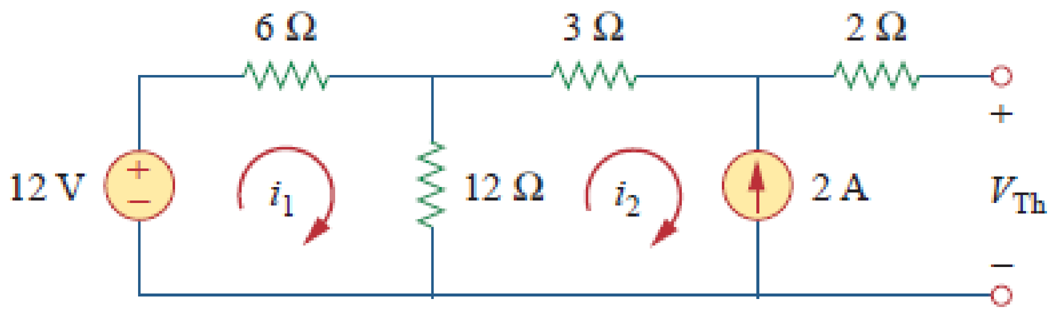
$$-12 + 6i_1 + 3i_2 + 2(0) + V_{Th} = 0 \qquad \Rightarrow \qquad V_{Th} = 22\text{ V}$$

For maximum power transfer,

$$R_L = R_{Th} = 9\ \Omega$$

and the maximum power is

$$p_{max} = \frac{V_{Th}^2}{4R_L} = \frac{22^2}{4 \times 9} = 13.44\text{ W}$$



(b)

EXAMPLE 9.17 Given the network in Fig. 9.88, find the value of R_L for maximum power to the load, and find the maximum power to the load.

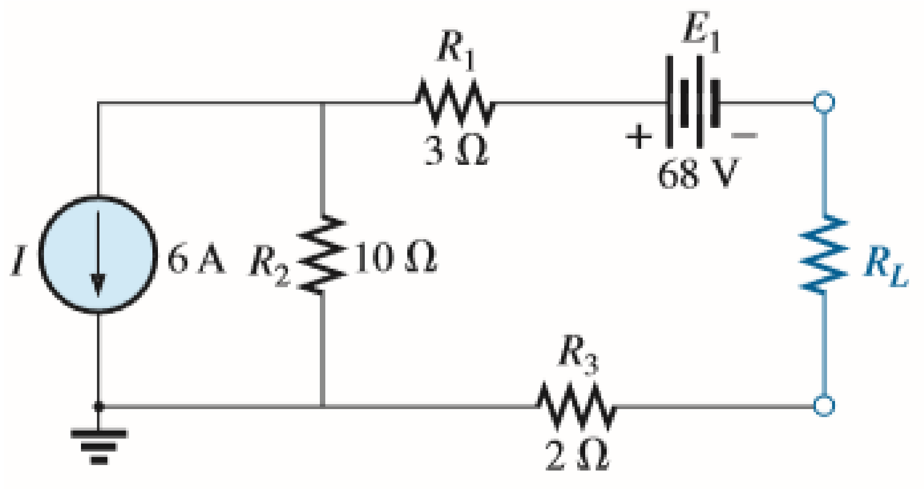
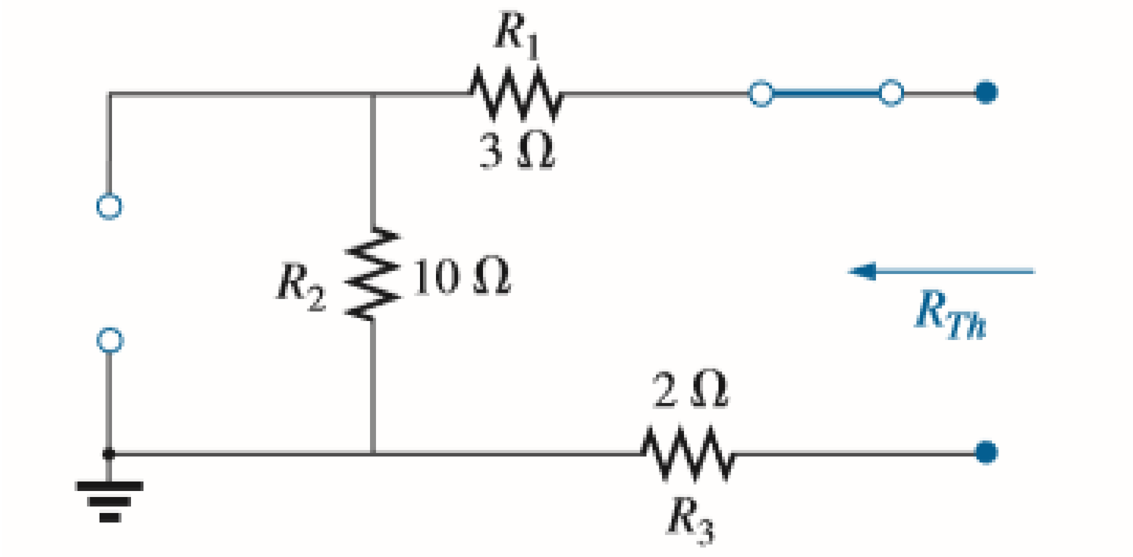


FIG. 9.88
Example 9.17.



$$R_{Th} = R_1 + R_2 + R_3 = 3\,\Omega + 10\,\Omega + 2\,\Omega = 15\,\Omega$$

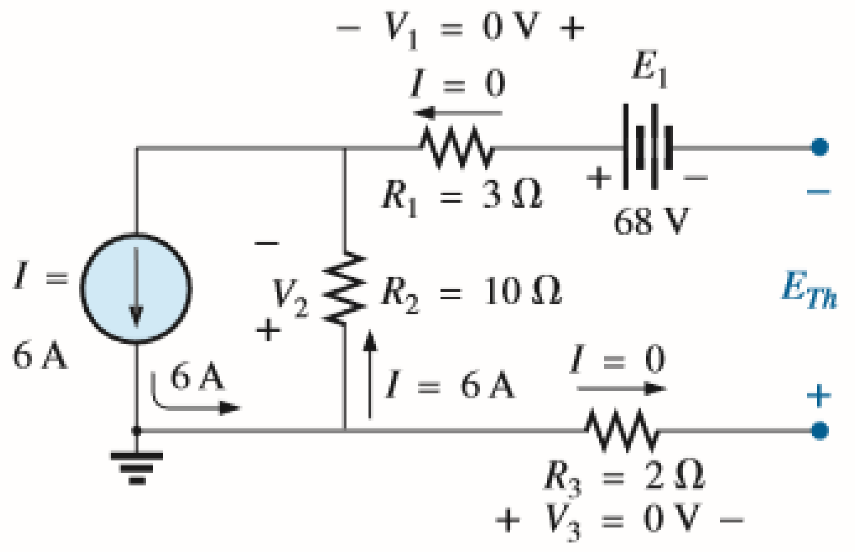


FIG. 9.90
Determining E_{Th} for the network external to resistor R_L in Fig. 9.88.

so that
$$R_L = R_{Th} = \mathbf{15\ \Omega}$$

The Thévenin voltage is determined using Fig. 9.90, where

$$V_1 = V_3 = 0\text{ V} \quad \text{and} \quad V_2 = I_2R_2 = IR_2 = (6\text{ A})(10\ \Omega) = 60\text{ V}$$

Applying Kirchhoff's voltage law:

$$-V_2 - E + E_{Th} = 0$$

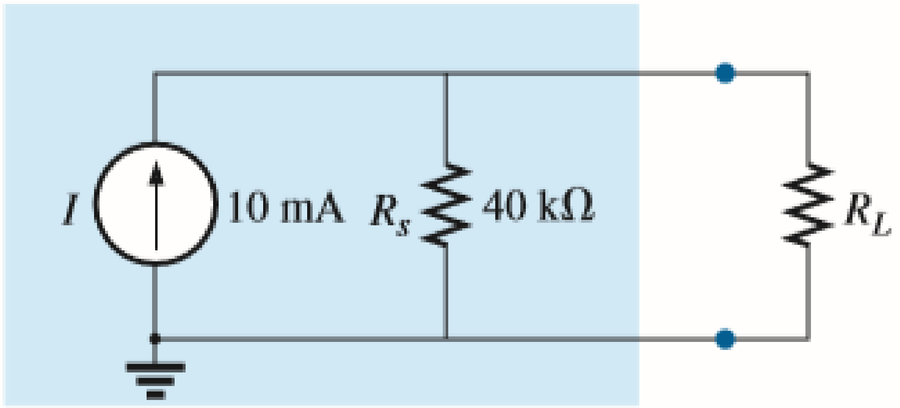
and
$$E_{Th} = V_2 + E = 60\text{ V} + 68\text{ V} = \mathbf{128\ V}$$

with the maximum power equal to

$$P_{L_{\max}} = \frac{E_{Th}^2}{4R_{Th}} = \frac{(128\text{ V})^2}{4(15\text{ k}\Omega)} = \mathbf{273.07\ W}$$

EXAMPLE 9.15 The analysis of a transistor network resulted in the reduced equivalent in Fig. 9.86.

a. Find the load resistance that will result in maximum power transfer to the load, and find the maximum power delivered.
b. If the load were changed to 68 kΩ, would you expect a fairly high level of power transfer to the load based on the results of part (a)? What would the new power level be? Is your initial assumption verified?
c. If the load were changed to 8.2 kΩ, would you expect a fairly high level of power transfer to the load based on the results of part (a)? What would the new power level be? Is your initial assumption verified?



a. Replacing the current source by an open-circuit equivalent results in

$$R_{Th} = R_s = 40\ \text{k}\Omega$$

Restoring the current source and finding the open-circuit voltage at the output terminals results in

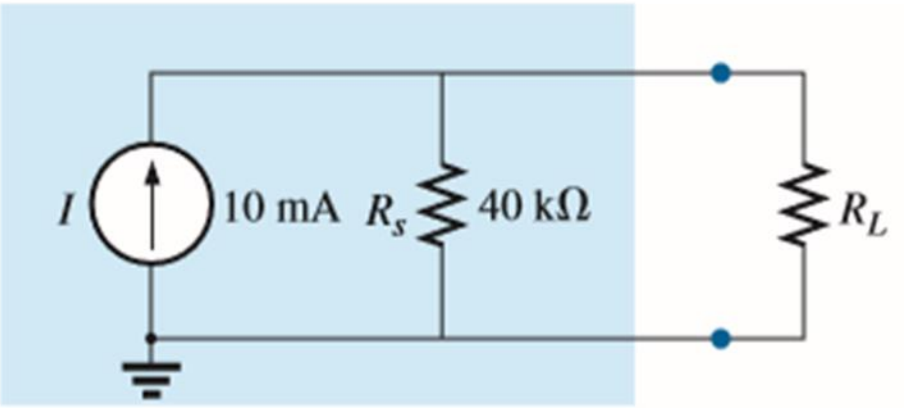
$$E_{Th} = V_{oc} = IR_s = (10\ \text{mA})(40\ \text{k}\Omega) = 400\ \text{V}$$

For maximum power transfer to the load,

$$R_L = R_{Th} = \mathbf{40\ k\Omega}$$

with a maximum power level of

$$P_{L_{max}} = \frac{E_{Th}^2}{4R_{Th}} = \frac{(400\ \text{V})^2}{4(40\ \text{k}\Omega)} = \mathbf{1\ W}$$



b. Yes, because the 68 kΩ load is greater (note Fig. 9.80) than the 40 kΩ load, but relatively close in magnitude.

$$I_L = \frac{E_{Th}}{R_{Th} + R_L} = \frac{400\text{ V}}{40\text{ k}\Omega + 68\text{ k}\Omega} = \frac{400}{108\text{ k}\Omega} \cong 3.7\text{ mA}$$

$$P_L = I_L^2 R_L = (3.7\text{ mA})^2(68\text{ k}\Omega \cong \mathbf{0.93\ W}$$

Yes, the power level of 0.93 W compared to the 1 W level of part (a) verifies the assumption.

c. No, 8.2 kΩ is quite a bit less (note Fig. 9.80) than the 40 kΩ value.

$$I_L = \frac{E_{Th}}{R_{Th} + R_L} = \frac{400\text{ V}}{40\text{ k}\Omega + 8.2\text{ k}\Omega} = \frac{400\text{ V}}{48.2\text{ k}\Omega} \cong 8.3\text{ mA}$$

$$P_L = I_L^2 R_L = (8.3\text{ mA})^2(8.2\text{ k}\Omega) \cong \mathbf{0.57\ W}$$

Yes, the power level of 0.57 W compared to the 1 W level of part (a) verifies the assumption.

Determine the value of R_L that will draw the maximum power from the rest of the circuit in Fig. 4.52. Calculate the maximum power.

Answer: 4.22 Ω, 2.901 W.

Practice Problem 4.13

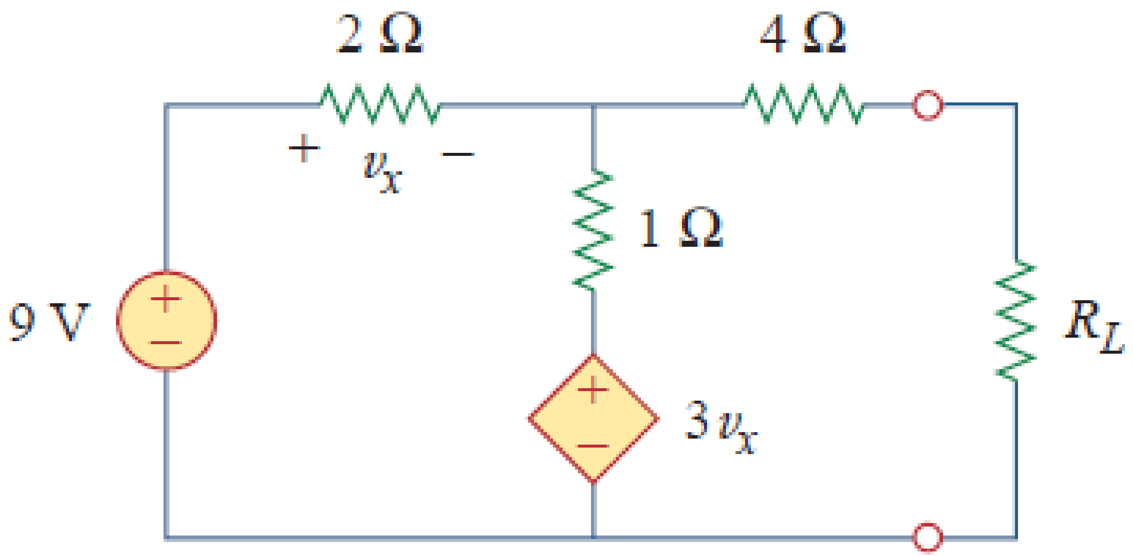


Figure 4.52
For Practice Prob. 4.13.

Thank You